

# STEM and Mathematics: Understanding and Developing Students' Fractions Knowledge through a Radical Constructivist Lens

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### Chapter Highlights

This chapter emphasizes the central role of mathematics within STEM education by positioning radical constructivism as a unifying theoretical framework applicable across STEM disciplines. It examines how mathematics education researchers employ radical constructivism to investigate learning processes, with a particular focus on fraction knowledge. Through an illustrative case study on children's understanding of fractions and pedagogical recommendations for supporting older students, the chapter bridges theory, research, and instructional practice.

- **STEM and Mathematics** – Examining the foundational role of mathematics as a core integrative component of STEM education.
- **Radical Constructivism as a Lens for STEM Education** – Introducing radical constructivism, contrasting it with the emergent perspective, and discussing its adaptation to classroom contexts.
- **Children's Fraction Knowledge** – Exploring how children develop an understanding of fractions, supported by an in-depth case study grounded in radical constructivist principles.
- **Older Students' Fraction Knowledge** – Offering instructional goals and pedagogical strategies to support and extend fraction understanding among older learners.

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## **Introduction**

As you can see in the chapter highlights, this chapter consisting of four subsections: (1) STEM and Math, (2) Radical Constructivism as a Lens for STEM Education, (3) Children's Fraction Knowledge, and (4) Older Students' Fraction Knowledge. Firstly, STEM education has been acknowledged as an essential core for K-12 and beyond as there are many jobs that require knowledge of STEM. Mathematics is a fundamental subject among STEM disciplines. This chapter starts with defining STEM education and the role of mathematics in STEM. Additionally, we suggest fractions as an exemplary topic in this chapter since fractions are used in all STEM disciplines as an essential concept that students need to understand conceptually. We believe this first section provides a broader view of STEM focusing on mathematics for readers.

Secondly, radical constructivism has been used as a framework for all disciplines of STEM. Radical constructivism concerns how a person constructs their knowledge based on their own experiences. In this second section, we explore how radical constructivism view can be applied for classroom context having more complex interaction between a teacher and multiple number of students. To investigate this, we compared emergent perspective to radical constructivism. Since emergent perspective emerged from radical constructivism, we focused on how the emergent perspective is applicable for classroom teaching. We expect this will give insight for teachers about how they can manage classroom discussion in a way that students learn mathematics connecting to their prior experiences or knowledge.

The third section concerns understanding and developing children's fractions knowledge through case-study. The second author of this chapter conducted a teaching experiment with a sixth-grade student to explore his fractions knowledge by posing six fraction problems. Teaching experiment is a methodology coming from radical constructivism aspect by focusing on understanding student thinking. We expect that this case study provides exemplary research to investigate student thinking focusing on a specific content. This approach can be applied for other STEM disciplines when they want to explore student thinking specifically in a topic.

In the final section of this chapter, we discuss older students' fractions knowledge since the topic is challenging for all ages based on the first author's teaching experiences in middle and high school and college teaching. This section suggests for educators to consider reorganization hypothesis, which appreciates students' prior experiences and knowledge of whole numbers

to construct new mathematical concepts of fractions. It also provides a brief overview of units coordination and schemes, which can help teachers understand their students' fractions knowledge to support them to develop from individual level of understanding. Finally, this section also suggests the usefulness of visual representation for teaching and learning fractions conceptually. For example, we included pictures that were created by a fraction bar tool, so that instructors can modify from it to apply for their students.

## **STEM and Math**

### **What is STEM education?**

STEM is an acronym for Science, Technology, Engineering, and Mathematics. The term “STEM” was introduced by the National Science Foundation (NSF) in the 1990s. Initially referred to as “SMET” (Science, Mathematics, Engineering, and Technology), the acronym was later changed to “STEM” due to its greater phonetic appeal and ease of use in communication. STEM education integrates four disciplines of science, technology, engineering, and mathematics to promote interdisciplinary learning, with an emphasis on hands-on activities, inquiry-based instruction, and problem-based learning. The goal is to develop students' critical thinking, creativity, and ability to apply knowledge to real-world challenges.

At the higher education level, Virginia Tech University is credited with launching the first graduate-level degree in STEM Education in 2005 to emphasize the educational dimensions of STEM training (Wells, 2013). The program includes core coursework such as EDCI 5804 – STEM Education Foundations, EDCI 5814 – STEM Education Pedagogy, EDCI 5824 – Trends and Issues in STEM Education, and EDCI 5834 – Research in STEM Education. Students may also take electives in areas such as biotechnology literacy (EDCI 5854) and complete field experiences (EDCI 5964) as part of their professional preparation.

In K–12 contexts, STEM education is understood in diverse ways depending on school settings and educator roles. A study identified three common themes across educators' views of STEM education: interdisciplinary connections among science, technology, engineering, and mathematics; the need for new, ambitious instructional practices; and student engagement in real-world problem solving (Holmlund et al., 2018). These shared elements highlight that STEM education involves integrating multiple disciplines through innovative teaching strategies that prepare students to address authentic challenges.

The study also found variation in how STEM education is conceptualized across different educational contexts and roles, suggesting that while a single, universal definition may not be necessary, it is important for educators within the same system to collaboratively develop a clear and common vision of STEM education tailored to their local context. This shared understanding can guide curriculum design, instructional methods, and professional learning to better serve students' STEM learning goals.

### **Why is math central to STEM education?**

Mathematics is widely recognized as the foundational discipline underpinning the other fields within STEM education. Its principles and methods provide the language and tools essential for understanding, analyzing, and solving problems across science, technology, and engineering.

In Science, mathematics is crucial for formulating hypotheses, designing experiments, interpreting data, and modeling natural phenomena. Quantitative reasoning enables scientists to express scientific relationships precisely and predict outcomes (Vera et al., 2021). A study identified prevalent misconceptions regarding mathematical modeling among biomedical experimentalists and suggested practical methods for addressing the cognitive distance between modelers and experimental researchers (Vera et al., 2021).

In Technology, mathematics supports the development and application of algorithms, data structures, and computational methods. From coding software to designing digital systems, mathematical concepts such as logic, discrete math, and statistics play a key role (Liu & Castellana, 2021). Additionally, in Engineering, mathematics provides the framework for designing, analyzing, and optimizing systems and structures (Ramkrishna & Amundson, 2004). Specifically, calculus, geometry, and linear algebra are vital for understanding forces, materials, and processes that engineers work with.

Despite its central role, the importance of mathematics in STEM education is often underestimated (Maass et al., 2019). This underappreciation can lead to insufficient focus on mathematical thinking and skills development, which are essential for success in STEM fields. Therefore, strengthening math education within STEM programs is critical to prepare learners for the interdisciplinary demands of STEM careers.

## **Why Fractions Matter: Foundational Reasoning for STEM Education**

Despite being introduced in elementary school, fractions remain one of the most conceptually challenging topics for students throughout K-12 and even into higher education. Research consistently shows that many students struggle to understand fractions as quantities, instead treating them as disconnected procedures or part-whole representations (cite). For example, Wiest and Amankonah (2019) emphasized that students often confuse the size of a fraction with the size of its components (e.g., thinking  $\frac{1}{5}$  is larger than  $\frac{1}{4}$  because 5 is larger than 4). Another study pointed out that relying heavily on procedures without conceptual understanding (e.g., cross multiplication with no sense of why it works) (Bansilal & Ubah, 2020). Also, Brown and Quinn (2006) found that students often face difficulties in interpreting improper fractions or mixed numbers. These challenges have long-term implications. Students who do not develop a solid understanding of fractions are more likely to struggle with algebra, proportional reasoning, and advanced mathematical thinking—all of which are foundational to success in STEM fields.

Fractions are not just a school subject—they are critical to reasoning and application across STEM disciplines. Specifically, in science, fractions support reasoning about proportional reasoning, which is foundational concept for concentration or density (Howe et al., 2011). For example, in chemistry, determining the concentration of a solution involves fractional relationship between solute and solvent. In biology, interpreting data such as population growth or decay often depends on fractional changes over time. In physics, concepts like velocity, acceleration, and density are expressed through ratios and fractional quantities. Therefore, without a firm grasp of how fractions operate, students may struggle to reason proportionally, estimate accurately, or interpret scientific data.

Engineering applications often relies on scaling and measurement, which require reasoning with fractions (Bird, 2014). For example, reading blueprints or creating scaled-down models of buildings or machines demands accurate fractional understanding. Moreover, material tolerances are often expressed as small fractional margins (e.g.,  $\frac{1}{8}$  of an inch). Structural load distribution and energy efficiency calculations also involve fractional reasoning. Hence, lack of fractions knowledge might hinder students' capacity to engage in authentic engineering design or prototyping.

In technology fields, fractions are embedded in both conceptual and

computational tasks (Fang et al., 2023). For instance, time-based media, such as editing audio or video, require fractional divisions of seconds (Herglotz et al., 2020). Also, data visualization often involves interpreting fractions, such as pie charts or percentages of user engagement. Especially in computer science, data encoding may involve fractional bases (Fang et al., 2023). Understanding fractions enables more efficient data manipulation and improved digital literacy.

Furthermore, fractions are fundamental to mathematical modeling, which is at the core of STEM problem-solving (Wilkins & Norton, 2018). For example, modeling with mathematics often involves fractional quantities, especially in real-world problems. Additionally, algebraic reasoning depends on students' comfort with variables and operations involving fractions (Hackenberg, 2013)2013. Thus, students' robust conceptual understanding of fractions plays a crucial role in access to STEM coursework.

### **Radical Constructivism as a Lense for STEM Education**

von Glasersfeld's (1995) radical constructivism serves as the underlying theory of learning. Rooted in Piaget's constructivism, radical constructivism holds that each student creates their understanding of the world based on prior experiences and interactions with the environment—which includes interactions with other people. Therefore, the teacher's role is guiding children to connect from their prior experiences to a new concept.

### **Emergent Perspective Takes from Radical Constructivism**

First, the Emergent Perspective “follows Glasersfeld (1992) in using the term *knowledge* in ‘Piaget’s adaptational sense to refer to sensory-motor and conceptual operations that have proved viable in the knower’s experience’” (Cobb, 2000, p. 154). Concepts of truth, viability, assimilation, accommodation, instrumentalism of knowledge, intersubjectivity, and reflective abstraction are also brought from Radical Constructivism. Similarly, communication between individuals is not seen as an exchange of fixed meanings. While communicating, to understand what other person has written or said, implies “to have built up a conceptual structure from an exchange of language, and, in the given context, this structure is deemed to be compatible with what the speaker appears to have had in mind(von Glasersfeld, 1995, p. 143).

In both perspectives, communication involves a process of negotiation of meaning, and interaction with others is conceived as an important source of perturbations. From Cobb's point of view, von Glasersfeld does

not only conceive that learning is stimulated by social interactions, but he conceives it as being social (Cobb, 2000). Therefore, both the Constructivist and Emergent Perspective consider learning as self-organization, socially, and culturally situated.

### **Differences between the Emergent Perspective and Radical Constructivism**

The divergence from Radical Constructivism started in 1986 (Cobb & Yackel, 1996). At the time, the authors were conducting *Developmental Research*, which they characterize as involving hand in hand, instructional development and classroom-based research. They initially used Radical Constructivism to interpret and discuss the students' mathematical conceptions, activity, beliefs, and learning in individualistic psychological terms. However, they concluded that such accounts were "inadequate" for the purposes of their developmental research study (p.176). As we interpret it, the individual accounts were inadequate in the sense that they were insufficient for describing the "students' mathematical development *as it occurs in the social context of the classroom*" (p.176).

That is, the result they obtained did not adequately account for the student's development in relation to what was happening at the level of the classroom micro-culture. Therefore, the divergence from radical constructivism was pushed forward by an experienced need to explore further the students' mathematical development in the social context of the classroom. This mathematical development is ultimately accounted for through the constructs: Classroom Social Norms, Socio-Mathematical Norms, and Classroom Practices.

From our interpretation, the central difference between one perspective and the other is that while Radical Constructivism focuses in the "individuals' construction of their ways of knowing" (Cobb, 2000, p. 155), the Emergent Perspective does not conceive of accounting for individual student's mathematical reasoning without accounting for the development of the classroom microculture. As they put it, they question "the assumption that such analyses [i.e., psychological constructivist analyses] can, in principle, capture individual students' conceptual understandings independently of situation and purpose" (p.185).

Instead, "individual student's mathematical activity and the classroom microculture are reflexively related (Cobb, 2000, p. 155). The relationship is deemed to be *reflexive* in the sense that the involved aspects are

interdependent, but neither can be adequately accounted for without considering the other. The individual thinking of the students contributes to the evolution of the classroom norms and practices, and the classroom norms and practices open and close down possibilities for students' learning. That is, both enable and constrain each other.

We previously mentioned that both perspectives conceive of social interaction as a rich source of perturbations and consequently, the result can be mathematical learning and development. Nevertheless, from Emergent Perspective's point of view, social interaction serves as more than just a *catalyst* of individual development. "Learning is not merely social in the sense that interactions with others serve as a catalyst for otherwise autonomous conceptual development. Instead, the products of learning, increasingly sophisticated ways of knowing, are also social through and through" (Cobb, 2000, p.154).

To close this section on differences between the two perspectives, we will make one last point. Studying classroom events by using the Emergent Perspective might involve similar processes as those proposed by Steffe and Thompson's (2000) constructivist teaching experiment (e.g., interviews, teaching episodes, on-going and retrospective analysis). Both intend to account for the students' individual reasoning and how they modify their mathematical activity.

One clear difference is that teaching experiments are usually conducted with one or a couple of students, whereas the Emergent Perspective is a framework that supports the interpretation of events either between a couple of students or during a full classroom lesson. Furthermore, besides accounting for individual learning, Emergent Perspective intends to parallelly account also for the mathematical development at the level of the social context of the classroom. In order to make these accounts is that Cobb and Yackel turned to Interactional Theory.

### ***Using Interactional Theory to Account for the Students' Mathematical Development as it Occurs in the Social Context of the Classroom***

As mentioned before, the Interactionist complement that constitutes the Emergent Perspective is mainly taken from Bauersfeld et al. (1988), work that was developed in light of Blumer's (1969) Symbolic Interactionism and Mehan and Wood's (1975) ethnomethodology. Blumer (1969) asserts that Symbolic Interactionism corresponds to a particular "approach to the study

of human group life and human conduct" (p.1). He explains that there are two levels of social interaction: *conversation gestures* and the *use of significant symbols* which correspond respectively with *non-symbolic interaction* and *symbolic interaction*. The difference between one and the other is that the second one involves an interpretation of action. While non-symbolic interaction is more evident, (e.g., in reflex responses), symbolic interactions require that the participants make an unobservable interpretation of each other's actions and utterances.

The interaction between a group of people consists of the fitting of the different participants' *lines of action*. This articulation gives way to a *joint action* that is not considered as the mere aggregation of each of the lines, and it is thought to be different from each of them. This joint action undergoes a process of formation and Blumer (1969) asserts that "even though it may be a well-established and repetitive form of social action, each instance of it has to be formed a new" (p.17).

Analogously, when referring to interactions in the context of a classroom, Cobb and Bauersfeld (1995) express that a "linguistic processes can be viewed as an accomplishment of language games that are special to each classroom, and in which the teacher and students negotiate *taken-as-shared* meanings and signs" (p.13). The expression 'taken-as-shared' is used to indicate that "individual interpretations fit for the purposes at hand but does not imply that they necessarily match" (Cobb, 1996, p.166).

In a math classroom, the interaction that occurs generates a negotiation of meaning in which each of the participants (i.e., the teacher and students), make adaptations and mutually establish expectations of each other. As these implicit negotiations take place, "teachers and students are seen to jointly constitute *classroom norms and practices* in the course of their interactions" (Cobb, 1996, p. 155). The norms and practices can be said to *emerge* from the interactions that occur among participants in the specific classroom context, which points particularly to the idea of "emergence" in the framework. Classroom practices are not considered to exist prior and independently from the teacher and student's activity; practices emerge through interactions of the teacher and the students.

In sum, Cobb and Yackel referred to Interactional Theory for developing the Emergent Perspective due to their shared view that "learning and understanding are inherently social and cultural activities" (Cobb & Yackel, 1996, p185). That is, student learning and understanding cannot

be accounted for without also accounting for the development of the classroom microculture. Note that although the Emergent Perspective shares many key assumptions and constructs with the Interactional Theory, Cobb and Yackel (1996) also point out some central differences between the Emergent Perspective and Socio-cultural Perspectives in general. Some of those differences include whereas the first focuses on a local community (i.e., a classroom microculture), the latter typically views individuals as participating in broader socio-cultural practices. Furthermore, while Socio-Cultural perspectives are often framed using conceptualizations such as negotiation as “mutual appropriation” and instructional issues as “transmission of culture”; the Emergent Perspective conceives negotiation as “mutual adaptation” and instructional issues in terms of “emergence” (Cobb & Yackel, 1996, p.186).

### ***A Closer Look at Using the Emergent Perspective in the Context of Developmental Research***

Now that the general ideas of the Emergent Perspective have been presented, we will provide more detail about the use of this perspective in the context of research. As mentioned above, the Emergent Perspective arose from a series of studies that involved *Developmental Research*. According to Cobb (1996), classroom analyses that are conducted through developmental research should fulfill three criteria: (a) They should emphasize not only the mathematical development of the classroom community, but also of the individual students. (b) The constant analysis performed should provide feedback with regards to the continuing process of instructional development. (c) The mathematical learning of the classroom community and the individual students should be documented in a detailed and meticulous way over extended periods of time.

In understanding what developmental research entails, we also find significant the way Gravemeijer (1994) describes it. That is, by citing Freudenthal, who in 1991 wrote that Developmental Research involves “experiencing the cyclic process of development and research so consciously, and reporting on it so candidly that it justifies itself, and that this experience can be transmitted to others to become like their own experience” (p. 452).

Cobb, Yackel and Wood (1992) further explain that Developmental Research requires a cyclical combination in which a *Development Aspect* is guided by discipline-specific instructional theory, and a *Research Aspect* is guided by an interpretative framework. On one hand, the *Developmental Aspect* alludes to classroom-based research used towards generating

instructional development. Existing literature is used to construct a sequence of instructional activities, which should envision how the teaching process and student's mathematical learning might proceed. Envisioning such a complex process involves formulating "conjectures about both student's possible learning trajectories, and the specific means of supporting, organizing and guiding that development" (Cobb, 1996, p.157).

Once a first version of the sequence of instructional activities is completed, its enactment is a necessary and important step. As Gravemeijer (1994) mentions, "what is invented behind the desk is immediately put into practice; [furthermore], what happens in the classroom is consequently analyzed, and the result of this analysis is used to continue the developmental work" (p. 449). In analyzing what is put into practice, the research team must analyze whether the activity proceeded as initially envisioned. Particularly by testing the formulated conjectures.

Besides enacting one or more instructional activities from the sequence in a classroom context, further insightful data can be generated from individual interactions with the students (e.g., interviews). The feedback obtained from the collected data allows researchers to reformulate their hypotheses and expectations, leading to modifying the instructional sequence, which should be put into practice and analyzed. In this sense the study undergoes a cyclical process that involves redesigning the instructional sequence and interpreting its implementation. Each of the iterations of the cycle involves constant reflections on the theoretical foundations and the empirical data produced. The cycle may act on different levels, between class sessions or between periods of time in which the research is being done.

On the other hand, the *Research Aspect* (i.e., interpretive framework) provides a set of assumptions, principles, and practices for the process of interpreting the generated data. Cobb and Yackel propose to use The Emergent Perspective to be such interpretive framework. In order to operationalize the framework in a way that makes its use accessible, Cobb and Yackel define *aspects* to account for from the psychological perspective and aspects to account for from the interactional perspective. Let's explore these aspects a little further.

### ***Aspects of Classroom Micro-Culture and its Psychological Correlates***

Providing rich and significant reports about both the classroom and individual development is no doubt a demanding but also insightful task.

To emphasize both kinds of development, the Emergent Perspective considers three main aspects of classroom microculture accompanied by three psychological *correlates* (see Table 1). The word correlate, as used in this perspective, alludes to a relation of complementarity, “a conjectured relation between an aspect of the classroom microculture and the activity of the individuals who participate and contribute to it.” (Cobb & Yackel, 1996, p. 177).

**Table 1.** Aspects of classroom microculture and the psychological correlates considered in the Emergent Perspective as interpretive framework.

| Aspects of classroom microculture | Individual aspects                                                                               |
|-----------------------------------|--------------------------------------------------------------------------------------------------|
| Classroom social norms            | Beliefs about own role, other’s roles, and the general nature of mathematical activity in school |
| Socio-mathematical norms          | Mathematical beliefs and values                                                                  |
| Classroom mathematical practices  | Mathematical conceptions and activity                                                            |

As mentioned before, classroom microculture and individual mathematical activity maintain a reflexive relation; therefore, the three pairs of corresponding aspects are also reflexively related. Neither of them is seen as being the cause of the other, and neither is claimed to come first. That is, for example, in the case of the first pair, individual beliefs contribute to the evolution of classroom norms and simultaneously, the renegotiation of social norms in the classroom supports the student’s reorganization of their beliefs. Although this reflexive relation is constantly emphasized in the articles, Cobb (1996) also expresses that they regard it as a conjecture open to empirical investigation.

**The Three Psychological Aspects.** Let’s address first the issue of how to account for the students’ individual progress. One thing that we have noticed is that in many different articles about the Emergent Perspective, not much is being said about the individual part of the analysis. We infer that this is because publications such as Cobb and Steffe (1983) and Steffe and Thompson (2000), among others, portrait rich explanations on how to carry out this type of process. That is, they usually include conducting individual (or with a small number of students) teaching experiments and interviews.

Nevertheless, our understanding is that what has been brought intact from constructivist teaching experiments is the use of interviews. As

suggested before, part of the data related to the evolution of the students' mathematical understanding and learning during the period of research, is obtained through a sequence of individual interviews. In these interviews, the main interest is to understand the students' reasoning, describe the changes in their procedures and strategies, and identify and explain the progress they have achieved.

We interpret that some other data to account for individual learning can also be obtained from the students' individual participation and contributions to the classroom discussions. From the examples provided in Cobb (1996), we also infer that an important part in accounting for individual learning is being aware of the different achievements of the students in comparison to other students or groups of students. Supporting evidence for this interpretation is the following description that Cobb (1996) uses when exemplifying individual progress:

"In contrast to the September interviews, ten of the eighteen students used non-counting thinking strategy solutions to solve all the tasks posed to them in interviews conducted in January... A further three students used to think strategies to solve at least half of the tasks presented, and the remaining five produces relatively sophisticated counting solutions." (p159)

From our point of view, the two other individual psychological aspects (i.e., beliefs about the students own role, the role of others and the nature of mathematical activity in school; as well as the mathematical beliefs and values), do not seem to have a primarily role in the descriptions and explanations presented as result of the research. What seems to be mostly documented is the communal organization of their beliefs, that is, the classroom mathematical practices and the socio-mathematical practices.

**The Three Aspects of Classroom Micro-Culture.** With regards to the three aspects of the classroom microculture, the collection and analysis of data resembles Steffe and Thompson's (2000) teaching experiments but transported to the classroom context. In this sense Cobb and Yackel (1996) refer to the process as constituting a *Classroom Teaching Experiment*. As previously indicated, central aspects to focus on when studying the development of the classroom community are social norms, socio-mathematical norms, and classroom mathematical practices. From our point of view, most of the articles about the Emergent Perspective dedicate a lot of more detail on these three social aspects because they constitute novel elements proposed by in the Emergent Perspective, in comparison to other interpretive frameworks. Let's explore each of these three aspects.

**Classroom social norms.** Cobb and Yackel (1996) note that when inquiring into a student's autonomous learning and development, one would need to account for their beliefs about their own role, others' roles and the general nature of mathematical activity in the school (i.e., the first individual aspect from table 1). The social correlation is classroom social norms, which primarily delineate classroom participation structure Cobb (1996).

From a general definition by the Merriam-Webster Dictionary (<https://www.merriam-webster.com>), a norm can be understood as "a principle of right action binding upon the members of a group and serving to guide, control, or regulate proper and acceptable behavior", and "a pattern or trait taken to be typical in the behavior of a social group". More specifically, Cobb and Yackel (1996) state that Classroom Social Norms "are not psychological processes or entities that can be attributed to any individual. Instead, they characterize regularities in communal or collective classroom activity and are jointly established by the teacher and students..." (p. 178).

Examples of these norms are the commitments that the students make within the classroom context, such as sharing and justifying their reasoning, listening and trying to understand their classmates' ideas, and being able to comment on them. Primarily therefore, Classroom Social Norms refer to acts of explanation, justification, and argumentation (Yackel & Cobb, 1996).

A particular case in which the first stages of emergence of social norms can be evidenced, is when a teacher starts to work with a new group of students. It is possible that the teacher's expectations about behavior and participation are different from the students' expectations and prior experiences (Cobb & Yackel, 1996). Consequently, a process of renegotiation of classroom social norms would be initiated. It is important to note that although the teacher is in charge of initiating, guiding and organizing the renegotiation process, the students also contribute to the evolution of the norms. Conversely, as the students interpret and make sense of the different contributions, they also reorganize their beliefs about the roles within a classroom.

A final point to make is that classroom social norms can also be found in classrooms from other subjects. Explanation, justification, and argumentation could easily be present outside of mathematics classrooms. Consequently, Cobb and Yackel found themselves with the need to differentiate these norms from those that involve mathematical elements. For this reason, another useful construct that constitutes the Emergent Perspective is the

socio-mathematical norms.

**Socio-Mathematical Norms.** These are “normative features of students mathematical activity”; that is primarily, norms of what counts as acceptable, different, or insightful mathematical solutions (Cobb, 1996, p. 161). The students with the teacher, through their discussion and interaction in class, decide whether a strategy is valid, if it is different from another already discussed, or if it is more sophisticated or desirable for the class. Cobb also explains that these norms evolve through time within a classroom, which strengthens the claim that mathematical norms can be very different from one classroom to another.

In developing this type of norm, it seems that initially the students might not be sure of what counts as acceptable, different, or more sophisticated answers. Not only the feedback, reactions and comments of the teacher help to the formation and evolution of the norm, but also the students' ideas and contributions. This is particularly evident for example, when the teacher instead of defining in advance a strictly unchangeable script of what will count as acceptable, different, or sophisticated; is open to make these decisions as a classroom community in the course of the interaction. The negotiation of socio-mathematical norms in this way is thought to allow the students to improve their mathematical reasoning and argumentation and avoid being limited to following a prescription from the teacher or the book.

Let us explore a little more what entails establishing norms regarding acceptable, different, and insightful mathematical solutions. First, in a discussion for deciding whether a solution is acceptable or valid, the students would have to share their strategies and procedures, while the rest of the class would have to interpret them. As the class reacts to the solutions and explanations, either with compliments, questions or objections, the classroom community negotiates what counts as an acceptable solution and what does not.

Second, establishing mathematical norms about similarities and differences is particularly stimulated when the teacher asks if anyone has solved the problem in a different way. Solutions could be judged to be different, for example, if they involved different calculational processes or different quantitative interpretations. These socio-mathematical norms emerge from the class discussion, and in Cobb and Yackel's (1996) classroom experiments, sometimes the judgement of the class was different from the judgment they, as researchers, made. The practice of comparing provides an

opportunity for the students to advance into higher-level cognitive activity.

Last, to decide whether an answer is more sophisticated or insightful than others, one essential element is the way the teacher responds to them. The physical reaction that the teacher has as well as expressions such as “yeah!” “I like that” “listen to her/him!” “that’s good!” provide elements that the classroom community uses to learn about what counts as “conceptually advanced forms of mathematical activity” (Yackel & Cobb, 1996, p. 465). These types of judgements are less evident in comparison to the previous two, since it is usually not explicitly discussed whether an answer was more satisfying or efficient than another, and “the children are left to decide in what sense the solution was special” (Yackel & Cobb, 1996, p. 464).

Cobb and Yackel (1996) indicate that a characteristic that should be met by socio-mathematical norms is that they result from mathematically productive discussions, which requires for the students to have developed “personal ways of judging.” (p. 179). During the negotiation of socio-mathematical norms, the students should not only explain and make arguments for their ideas but also take the explanations as objects of reflection. They must make their explanations not only valid for themselves, but also understandable for their classmates.

There will always exist individual differences and achievements from one student to another. It is part of the teacher’s role to be aware of the individual advances of his/her students and to accept and value different types of answers based on that. The teacher plays a very challenging role, since she or he is in charge of legitimizing the students contributions and also, “it is the teacher’s responsibility to make judgments about the extent to which students take something as shared and to facilitate communication by explicating the need for further explanation” (Yackel & Cobb, 1996, p.471).

When reflecting about social norms and mathematical norms, it is possible to find oneself with many questions about what counts as a norm and what does not. For example, if a classroom social norm is that all the students should try to understand their peers’ ideas, does that mean that all the students do understand all of their peers’ ideas all the time? We would say that the answer is no. A reason why trying to understand other students’ ideas might count as a norm could be its quality of happening regularly.

A more complex question could be, how can one determine regularity? Or, what if some students act according to the norm, but other students do

not at all? Although we have found explicit reference to questions of this type. The establishment of a norm is highly context- dependent, reason for which, we infer, it is not possible to offer an exact set of parameters that define what counts and does not count as a norm. We further infer that a norm is a taken-as-shared understanding of what is expected and possibly encouraged in the classroom micro-culture. That is, a student might not always act according to a specific norm, and still consider it as something that is expected from him/her to engage in.

One last key question that might arise could be related to the boundary line between classroom social norms and socio-mathematical norms. We personally think that a good way to differentiate between one and the other is thinking that the first type of norm leans towards ways of participation in the class (e.g., explaining procedures, judging, and classifying strategies). The second type of norm, on the other hand, refers to the products of the reflections and discussions that occur in the classroom. An example of a socio-mathematical norm could be, when subtracting  $11-7$ , conceiving (a) subtracting seven from ten and adding one, as more sophisticated than (b) counting forward from seven to eleven. The act of discussing and judging if they are valid or different, is a classroom social norm; but the taken-as-shared agreement that the type of strategy (a) is more sophisticated than the type (b), is a socio-mathematical norm.

***Classroom Mathematical Practices.*** The third and last element to account for when studying the development of a classroom community is the Classroom Mathematical Practices. It is important to note that the psychological correlative of this aspect is the student's mathematical conceptions. In a constructivist teaching experiment, as described by Cobb and Steffe (1983), one of the goals is to identify and explain patterns and evolution of the students' reasoning. The Classroom Mathematical Practices also support tracing the mathematical development and the evolution of the student's procedures and reasoning as they become more sophisticated. The difference is that the practices concern procedures and reasoning of the classroom community as a whole. In Cobb (1996) and Cobb and Yackel (1996), the classroom mathematical practices are described as practices that become routine, practices that do not need justification, and interpretations that are *taken as shared* by the teacher and the students. To understand better what it is referred to when saying that a practice becomes routine, suppose that a student proposes a novel, appealing, or effective new way of solving a specific kind of problem. At the beginning, the classroom community might have an active discussion about the acceptance and qualities of this

practice. The students that propose the practice might need to provide clear justifications for its acceptance. Eventually, the classroom community might continue to use the practice, until it becomes a common routine, and does not need further justification. What started as a novel way of solving a problem would become classroom mathematical practice.

Not that it is not assumed that an accepted practice is one that is understood in the same way by everyone in the classroom community. The *taken as shared* interpretations and practices are said to be established as opposed to *shared* interpretations and practices. A taken as shared interpretation of the practice entails attaining intersubjectivity (cf. Von Glasersfeld, 1995), in the sense that there are no apparent disagreements present.

Cobb (1996) explains that these practices generally evolve and change over time. As an example, consider the following evolution of a classroom mathematical practice for adding a group of six candies to a group of four candies. At first, the class could have as a routine to start counting all the first six candies (i.e., one, two, three..., six) and then continue counting (i.e., seven, eight, nine, ten) until finding the answer. Later on, the class could migrate to start from six and count forward (i.e., seven, eight, nine, ten) to find the answer. Eventually, the class could agree to use more sophisticated strategies such as transferring one candy from the second group to the first one, obtaining five plus five and a more immediate response of ten. It is worth saying that this example was not taken from a documented real experience, but it is a product of our personal understanding of what counts as an example of mathematical practices.

Similar to what We exposed for the norms; one can encounter several key questions when reflecting about mathematical practices. For example, in the candy example, transferring a candy from one group to the other is a significantly more advanced way of solving the problem than counting the first six candies and then continuing counting. In a classroom, there could be students at different stages in their numerical development. If some students can easily conceive transferring a candy from one group to the other, but other students do not yet reason in this way, can the transferring strategy be conceived as a classroom mathematical practice?

Although we did not find an explicit response to questions like this, we infer that the response is yes. We infer that establishing a mathematical practice does not necessarily mean that every single student in the classroom

necessarily engages in the practice. From our interpretation of both Cobb (1996) and Cobb and Yackel (1996), we understand that it is possible that multiple practices are enacted at the same time, depending on the individual development and preferences of the students. That is, in the context of the candy problem, it is possible that some students use the transferring strategy, whereas other students remain using the counting from six strategy. Both practices considered established and accepted within the classroom community. Furthermore, for the latter students, the transferring strategy could represent a practice to work towards.

Another aspect to point out is that a potential shift from one practice to another might occur over a period of several weeks. Although the students contribute to the emergence of the practice, the teacher, as leader in the classroom, takes an important part in pursuing the establishment of a particular practice. Usually, mathematical practices emerge from explicit negotiations facilitated by the teacher. This brings forth connections between the mathematical practices and the learning goals that the teacher determines for the lesson or unit. In this sense, Cobb and Yackel (1996) point out the relevance of mathematical practices towards documenting instructional sequences that take place in classroom interaction, as well as documenting the social situations in which students participate and learn. Ultimately, portraying the process of the mathematical development of the classroom community.

Cobb and Yackel (1996) stress the clarification that making accounts of the mathematical development of the classroom as a community does not deny or ignore the student's individual differences. It is the goal of the psychological analysis aspect of the Emergent Perspective (i.e., individual aspects shown in table 1), to "reveal qualitative differences in individual children's mathematical interpretations even as they participate in the same mathematical practices" (p.180). At the end of successful developmental research using the Emergent Perspective, the researchers should be able to differentiate students from different classrooms, and they should also be able to identify differences between groups of students and individual students within a classroom. This would be a manifestation of having accounted for both classroom micro-culture and individual development.

### **A Case Study of Understanding and Developing Children's Fractions Knowledge**

For much of the history of mathematics education, the subject has often been treated as a fixed body of unquestionable rules and procedures to

be mastered through repetition and memorization. This traditional view positioned learning as the accumulation of established facts rather than as an active process of meaning-making. In recent decades, however, mathematics education has undergone significant change, influenced by scholars and educators who view mathematics through multiple perspectives. Questions such as “How can mathematics be taught most effectively?” and “How can we engage children’s genuine interest in mathematics?” have driven efforts to renew both theory and practice (Nelissen, 1999). As part of this evolution, earlier learning theories have been refined, and new ones have emerged.

One of the most influential is constructivist learning theory, often described as a “learner-centered” or “interest” theory. In mathematics education, constructivism emphasizes that children actively build mental representations images, diagrams, methods, intuitions, and thought processes in response to mathematical ideas. Teaching mathematics through constructivist activities means valuing these representations and the discoveries children make, positioning the learner’s thinking at the center of instruction.

The belief that learning is an active process undertaken by the learner, rather than the passive reception of knowledge, can be traced to philosophers such as Socrates and Kant (von Glasersfeld, 1991). In education, constructivism’s most significant development came through the work of Jean Piaget (1896–1980), founder of cognitive psychology, whose research shaped contemporary understandings of how knowledge is formed (von Glasersfeld, 1991).

In Turkey’s national curriculum, the constructivist approach has been adopted as a guiding principle for mathematics education. Teachers are expected to use student-centered techniques, strategies, and methods that allow learners to discover and make sense of concepts independently. Teacher training programs and school curricula are aligned with this philosophy.

The present study had two main aims: (1) to understand the fraction knowledge of a child taught under a constructivist-oriented curriculum, and (2) to develop the child’s knowledge of multiplying fractions, a topic he/she had not yet formally studied.

### **Background on Fraction Learning Challenges**

Fractions have long been a central yet challenging topic in mathematics education. Many students struggle with fractions well into secondary and

even post-secondary education. In some cases, proficiency with fractions is considered a predictor of broader mathematical success (Kerslake, 1986). Yet fraction instruction often induces anxiety, sometimes as a result of overly procedural teaching. When fractions are taught only as a set of rules—such as “With the same numerator, the fraction with the smaller denominator is greater” or “When dividing by a fraction, invert and multiply” students may fail to understand the concepts underlying those rules. As a result, they often misapply procedures and struggle to connect fractions with their natural number knowledge.

### **Theoretical Framework**

This case study draws primarily on Steffe and Olive (2010) as a reference for understanding children’s fraction knowledge. A central hypothesis in their work is that fraction knowledge develops in harmony with, and is constrained by, a child’s existing whole number understanding. Rather than relying solely on adult or formal mathematical structures to interpret a child’s thinking, Steffe and Olive focus on what children themselves can construct as mathematics.

To examine fraction knowledge, it is important to distinguish between first-order and second-order mathematical knowledge (Cobb & Steffe, 2011): First-order mathematical knowledge refers to the mental models and structures an individual constructs to organize and make sense of their own experiences. Second-order mathematical knowledge refers to the models that an observer (such as a researcher or teacher) constructs to describe and interpret another person’s mathematical understanding.

Steffe and Olive (2010) were primarily concerned with second-order knowledge, arguing that to assess children’s mathematical understanding accurately, educators must focus on the child’s own mathematics not merely on adult interpretations of it. They use the terms: “Children’s mathematics” as whatever constitutes a child’s first-order mathematical knowledge, “Mathematics of children” as second-order models of children’s mathematics, and “Mathematics for children” as concepts and operations that children are ready to learn, based on their current understanding. In building “mathematics for children,” understanding both children’s mathematics and the mathematics of children serves as the foundation for instructional design.

## **Number Sequence Development**

Research on children's number sequences further informs this study. From teaching experiments, Steffe and colleagues identified three developmental stages: (1) Initial Number Sequence (INS) internalization of basic counting activity, (2) Tacitly Nested Number Sequence (TNS) recognition of parts within a whole without explicit coordination, and (3) Explicitly Nested Number Sequence (ENS) explicit coordination of nested units within a whole.

Each new sequence is built upon the restructuring of the previous one, and gaps in this progression can hinder later learning. For example, a child lacking fluency with whole numbers may also struggle to develop robust fraction concepts. Conversely, mastery at one stage supports readiness for the next. In this study, the student had progressed beyond ENS to the Generalized Number Sequence (GNS) stage, enabling efficient whole-number operations but still showing difficulty linking those skills to fraction reasoning.

## **Methodology**

The study involved interviews with a sixth-grade student, "Bernard," using six fraction problems (Table 2). The first four questions probed his existing fraction knowledge; the last two were presented after instructional sessions on multiplying fractions. Data were collected via Zoom, with both video and written work recorded.

During each task, the teacher used the Fractions Bars software to mirror and clarify the student's representations. The instructional sequence incorporated three lessons from a Turkish Ministry of Education-approved textbook, supplemented with explanations addressing misconceptions revealed in the first four problems. Notably, Bernard reported that his school lessons did not use computer-based tools, instead relying on a locally published textbook.

**Table 2.** Focus and Tasks for Data Collection

| Focus                                           | Task                                                                                                                                                                                                                                                                |
|-------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Student's prerequisite knowledge of fractions   | Q1. Enes divides the first stick in his hand into 4 pieces and takes one for himself. Then he divides the second rod of the same length into 5 parts and takes one for himself. Which piece is longer?                                                              |
|                                                 | Q2. Bekir divides a stick in his hand into 5 equal parts and paints them blue. Then he divides another stick of the same length into 5 equal parts and paints three of them green.                                                                                  |
|                                                 | Q3. Bekir divides a stick into 4 equal parts and paints 3 of them blue. Then he divides another stick of the same length into 7 equal parts and paints 5 of them green.                                                                                             |
|                                                 | Which one is longer? The total length of the green bars or blue bars?                                                                                                                                                                                               |
| Researcher-led teaching fraction multiplication | Q4. We have two rectangular cakes of equal size. How would you share these two cakes equally among 3 people named Bekir, Enes, and Ashly?                                                                                                                           |
|                                                 | What is the ratio of the cake would Bekir get and one cake?                                                                                                                                                                                                         |
|                                                 | What is the ratio of the cake would Enes get and two cakes?                                                                                                                                                                                                         |
|                                                 | Q5. Bekir reserved $\frac{1}{3}$ of the cake for himself. Later, he divided the cake he had allocated for herself into 7 equal parts and took 1 portion for himself. In the last case, what is the ratio of the cake Bekir allocated for himself to the whole cake? |
|                                                 | Q6. Enes answers $\frac{5}{6}$ five-sixths of the questions in the exam. Since $\frac{2}{3}$ two-thirds of the questions answered by Enes are correct, what is the ratio of the questions Enes answered correctly to all questions?                                 |
|                                                 |                                                                                                                                                                                                                                                                     |

## Analysis and Protocols

To assess Bernard's prior understanding of fractions, we analyzed data from the first four interview tasks. Each problem explored a different aspect of fraction comparison, representation, and reasoning. Because Bernard had been educated entirely in Turkey and spoke only Turkish, all interviews were conducted in Turkish and later translated into English by the author. The names in the problems were chosen from familiar Turkish names to help him visualize the scenarios more easily. This strategy, informed by our earlier teaching experience, helped reduce student anxiety and bias toward word problems.

A recurring challenge for many learners is that fear of mathematics can create a barrier to performance. Building a safe and supportive learning environment where students feel comfortable making mistakes is critical to helping them engage fully with mathematical ideas.

## **Protocol 1 – Comparing Different Partitions of Equal Wholes**

Task: Enes divides the first stick into 4 equal pieces and keeps one. He then divides a second stick of the same length into 5 equal pieces and keeps one. Which piece is longer?

Student's Work and Dialogue:

Bernard first drew the two sticks. He divided the first into four equal parts and the second into what he initially thought were also four parts. Upon correction, he redrew the second stick into five equal parts.

Researcher: Which piece is longer?

Bernard: The first piece.

Researcher: Why?

Bernard: It's  $\frac{1}{4}$ . (pause)

Researcher: And the second piece?

Bernard: Five over one... oh, no—it's  $\frac{1}{5}$ .

Researcher: Why is the first longer?

Bernard: Because the square made by  $\frac{1}{4}$  is larger than the one made by  $\frac{1}{5}$ .

Interpretation:

This task assessed understanding of how partitioning affects unit size. Bernard confidently identified the longer piece and explained the relationship between denominator size and part size. His reasoning demonstrated that, for equal numerators, a larger denominator produces a smaller fraction. Although the task was relatively simple for his grade level, it set the stage for more complex comparisons in subsequent protocols.

## **Protocol 2. Comparing Fractions with Equal Denominators**

Task: Bekir divides one stick into 5 equal parts and paints two parts blue. He divides another stick of the same length into 5 equal parts and paints three parts green. Which total is longer: the green parts or the blue parts?

Student's Work and Dialogue:

Bernard drew both sticks, dividing each into five parts. He colored two parts of the first stick blue and three parts of the second stick green.

Bernard: Green is longer, but blue is bigger.

Researcher: How is that?

Bernard: There are 3 green parts and 2 blue parts... but... 2 is bigger

because 3 remain yellow.

Researcher: Let's write them as fractions. Blue?

Bernard:  $2/5$ .

Researcher: Green?

Bernard:  $3/5$ .

Researcher: Which is longer?

Bernard: (pause)  $3/5$ . Greens are longer.

Interpretation:

Initially, Bernard confused the total number of colored parts with the size of the parts themselves, suggesting a momentary mix of quantitative and qualitative reasoning. Once the fractions were expressed numerically, he correctly concluded that  $3/5$  is longer than  $2/5$ . His success here, as in Protocol 1, indicated secure understanding of comparisons when denominators are equal.



**Figure 1.** Bernard's Picture and Teacher's Picture

### **Protocol 3. Comparing Fractions with Different Denominators**

Task: Bekir divides one stick into 4 equal parts and paints 3 parts blue. He divides another stick of the same length into 7 equal parts and paints 5 parts green. Which is longer: the total green length or the total blue length?

Student's Work and Dialogue:

Bernard drew the first stick in the fourth, coloring three parts blue, and the second stick in sevenths, coloring five parts green.

Researcher: Fractions for each?

Bernard: Blues are  $3/4$ , greens are  $5/7$ .

Researcher: Which is longer?

Bernard:  $5/7$ —because the numbers are bigger.

Researcher: What if we look at them side by side?

Bernard: Oh...  $3/4$  is longer.

Researcher: Do you remember what to do when comparing fractions

with different denominators?

Bernard: I don't remember.

Bernard's first response relied on comparing numerators and denominators in isolation rather than in relation. His visual model was also inaccurate, the two drawn sticks were not the same length, making the comparison harder. When presented with an accurate visual model in the Fractions Bars program, he recognized the correct answer. The exchange revealed gaps in his procedural knowledge for comparing fractions with unlike denominators, suggesting that he had either not fully learned or had forgotten how to find common denominators.

#### **Protocol 4. Partitioning Multiple Wholes Equally**

Task: Two rectangular cakes of equal size must be shared equally among three people: Bekir, Enes, and Ashly.

What fraction of one cake does Bekir get?

What fraction of two cakes does Enes get?

Student's Work and Dialogue:

Bernard drew two identical rectangles, each divided into three equal parts. He assigned two parts from each cake to each person.

Researcher: How many pieces does each person get?

Bernard: Two pieces. That's  $\frac{2}{3}$ . Everyone gets  $\frac{2}{3}$ .

Researcher: So Bekir gets what fraction of one cake?

Bernard:  $\frac{2}{3}$ .

Researcher: And Enes from two cakes?

Bernard:  $\frac{1}{3}$ —because he gets  $\frac{1}{3}$  of each cake.

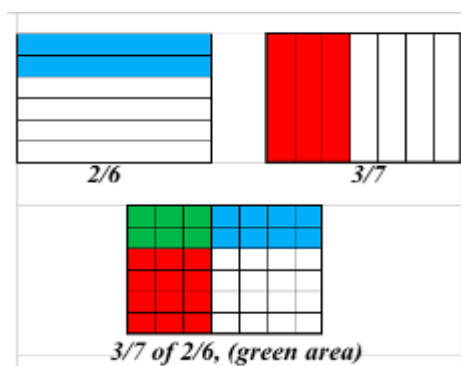
This problem assessed his ability to coordinate multiple wholes in an equal-sharing context. Bernard's correct and confident reasoning suggested he could operate at a level corresponding to at least Stage 3 of Hackenberg's fractions framework, managing composite units across multiple wholes without difficulty. In the next stage, the lessons are made with Bernard and as a result of these lessons, the development of Bernard on multiplying fractions will be examined. Our reference source for lecturing has been the (Hackenberg et al., 2016) book, and the examples in this book were used during the lecture.

#### **Developing Children's Fraction Knowledge**

As a teacher with prior experience working mainly with high school

students, we had limited experience teaching middle school learners. Before working with Bernard, we watched numerous instructional videos from online course platforms commonly used by students. We also drew on examples from *Developing Fractions Knowledge* (Chapter 10) by Hackenberg et al. (2016) to inform our lessons. One observation stood out: many online platforms focus almost exclusively on teaching rules when introducing fractions. In these lessons, the emphasis is on memorizing a procedure rather than allowing students to discover it for themselves a clear departure from the constructivist approach. Almost all teachers began with statements such as, "When you multiply fractions, first multiply the numerators, then multiply the denominators, and write the result as a fraction."

This teacher-centered approach raises questions about how truly student-centered such instruction can be. In contrast, the mathematics textbook recommended by the Turkish Ministry of National Education employs more student-centered methods, such as the modeling method. In our work with Bernard, we relied on this approach to help him discover concepts for himself. Initially, we introduced taking unit fractions of a unit fraction and taking non-unit fractions of unit fractions using fraction bars. However, representing more complex fraction relationships visually proved challenging with fraction bars alone. For problems involving fraction multiplication, we used the modeling method (see Figure 2).



**Figure 2.** Modelling Method

In addition, we addressed fraction simplification so that Bernard could recognize when simplification was needed especially in problems involving larger numbers. We believe that procedural rules can be helpful for efficiency, but they should be introduced after students have developed the underlying concept for themselves. Our goal was not only for Bernard to be able to solve

fraction problems correctly, but also to be able to model them independently and apply his understanding when faced with more complex tasks in the future. Even though sixth-grade students may not typically encounter highly complex fraction problems, a strong conceptual foundation at the primary and middle school levels has long-term benefits for future learning.

After instruction, we presented two interview problems to assess Bernard's progress with fraction multiplication one focusing on unit fractions of a unit fraction, and the other on general fractions of fractions.



**Figure 3.** Bernard's Picture

### **Protocol 5**

Question: Bekir reserved  $\frac{1}{3}$  of a cake for himself. Later, he divided that portion into 7 equal parts and took 1 part for himself. What is the ratio of the cake Bekir allocated for himself to the whole cake?

Student's Process:

Bernard drew a rectangular cake and divided it into three equal parts. He then divided one of those parts into seven equal sections.

Researcher: What fraction of the whole cake did Bekir keep for himself?

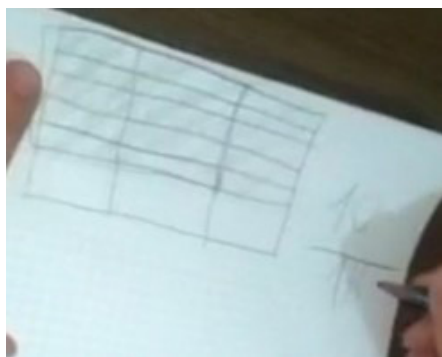
Bernard:  $\frac{1}{21}$ .

Researcher: How did you get 21?

Bernard: Each piece has 7 parts. Three pieces together make 21 parts, so it's one part out of 21.

This problem involved taking a unit fraction of a unit fraction. While a student who is already comfortable with natural number multiplication could quickly be shown the rule for multiplying fractions, we chose instead to let Bernard discover the relationship himself. He used fraction bars (see Figure 3) to model the problem step-by-step, arriving at the correct answer without difficulty. This approach allowed him to generalize the idea for more complex situations. Students taught only procedural rules often struggle with problems presented in unfamiliar formats, whereas Bernard's

conceptual understanding helped him solve the problem with confidence.



**Figure 4.** Bernard's Picture

### **Protocol 6**

Q6. Enes answers  $(5/6)$  five-sixths of the questions in the exam. Since  $(2/3)$  two-thirds of the questions answered by Enes are correct, what is the ratio of the questions Enes answered correctly to all questions?

Bernard: The student read the question and started. First, he drew a rectangle and divided it horizontally into 6 parts. Then he divided it into 3 parts vertically by the Modeling method. T- What have you done so far?

Bernard: First I drew five-six, then two thirds. The areas that were painted together are important to us, we will count them. There are 10 we are writing here. All of them have 18 in total. So here we find that  $2/3$  of  $5/6$  is  $10/18$ .

Researcher: Ok, thank you.

It was difficult for the student to reach a solution by using the fraction bars program in fractions with different numerators and denominators, and it was not successful in reaching a solution. Therefore, as we asked in Protocol 6, the student's use of the modeling method in questions helped him explore better. Bernard chose the table we used in the modeling method as in Figure 4 when solving the question and thus did not have difficulty solving the question.

This problem required multiplying fractions with unlike denominators. In this case, using fraction bars was less effective, so Bernard chose the modeling method. This visual strategy helped him work through the problem systematically. He demonstrated clear understanding of both the concept and the procedure, and his solution was correct and simplified.

Across the two protocols, Bernard consistently demonstrated progress in conceptualizing fraction multiplication. In Protocol 5, he was able to model a unit fraction of a unit fraction using fraction bars, reasoning his way to the correct answer without relying on memorized procedures. In Protocol 6, he successfully applied the modeling method to a problem involving unlike denominators, again reaching the correct and simplified solution.

These findings suggest that introducing multiplication of fractions through discovery-based methods—such as fraction bars and the modeling method—can help students build strong conceptual foundations before learning formal rules. Once these rules are introduced, students can connect them to the underlying meaning, making them more adaptable to non-routine problems. In Bernard’s case, the gradual transition from visual models to symbolic rules seemed to support both understanding and fluency.

### Older Students’ Fractions Knowledge

In a first author’s experience in teaching math for 11th graders in Ethiopia, she faced some students who had difficulty in computing fractional numbers. For example, she was teaching about the equation of ellipse, which is in the curriculum for 11th grade. After class, one of her students asked her about a line of calculation while the student said he understands how to find the equation of ellipse. She reminded that was “ $5-9/2=1/2$ ” because it was surprising for her that an 11th-grade student could have difficulty in the computation of fractional numbers, not the equation of ellipse, which is their challenging task in their grade. These experiences led her to study more in the Mathematics Education field in order to find the reason and solution that can help those students. That example is one of her experiences. We had more students having difficulty in calculating fractional numbers, even though they learned fractional numbers in elementary school.

Through her experience, we came up with four ideas for reasons why older students have difficulty in fractional knowledge and how to figure them out. We investigated those reasons and solutions with reference books and articles of Steffe and his students. Four ideas are as below:

1. In view of *reorganization hypothesis*, students might reorganize their fraction scheme based on previous arithmetic knowledge.
2. Many students have difficulty in understanding improper fractions. *Units coordination* could be a key for learning improper fractions knowledge.
3. Teachers need to know their students’ *multiplicative concepts*. And

then, they are able to provide learning by corresponding level for more understanding fractions knowledge.

4. Many students learn fractional computation *procedurally* rather than visually. Thus, students do not understand fractions as *concept images* which includes all the mental pictures and associated properties and processes (Tall & Vinner, 1981), so it is easy to forget how to apply for solving questions.

In this paper, we respond to four ideas to understand the reason why older student's difficulty in fractions knowledge have even if they learned in their elementary school. To do so, we will try to find evidence of students' understanding fractions knowledge and suggestions for improving their fractions knowledge based on four goals.

From the authors' internship at undergraduate course of N101, Teaching and Learning Elementary School Mathematics, we experienced how to approach the reasoning of operation in natural numbers and fractions with strategies. Most of the strategies were expressed as drawing and students were needed to draw by themselves every class. It was very new for us because basic operations such as addition, subtraction, multiplication, and division are too basic to draw for me, so we had limited experiences to draw them. However, we realized visualizing and explaining strategies by means of students' words are helpful for students to understand operations and even helpful to understand fractional knowledge deeply.

Students have an image of fractional numbers as part of a whole, so they have difficulty understanding improper fractional numbers because it needs to know improper fraction could be beyond a whole. In Hackenberg and Lee's (2015) 2015 research, a student figured out improper fractions by using the concept of refill of cake. It infers, in specific context, students can figure out the meaning of improper fractions visually. This kind of example can be a clue to starting to understand improper fractions. We believe that visualizing fractional knowledge with specific object can be helpful to have fractions concept image for children before enabling abstract thinking of fractions. And units coordination can be helpful for students to interiorize their fractions knowledge by visualizing fractions as well as imagining fractions with mental actions. In this chapter, we demonstrated how to interiorize fractional knowledge and make a scheme by using units coordination.

## Reorganization Hypothesis

The basic hypothesis in this paper is the *reorganization hypothesis*. Steffe

and Olive (2010) coined that one could consider the new way of knowing to be a rearrangement of the prior way of knowing when it is built using the prior way of knowing in a creative way. The *interference hypothesis*, which is a widely held opinion that natural number knowing interferes with fractional knowing, stands in contrast to the reorganization hypothesis (Steffe & Olive, 2010). As a result, when children answer fractions problems, they are seen to operate on natural numbers in a manner that is inconsistent with their operations on fractions.

Children can construct their fractions scheme by eliminating *perturbation* by making *accommodation*. *Perturbation* is defined as either disappointment or surprise and it may lead to all sorts of random reactions (von Glasersfeld, 1995). Accommodation is an act of learning in two cases. Firstly, if the activity's unanticipated result disappointed you, one or more of the recently observed traits might alter the recognition pattern and, consequently, the circumstances that will set off the activity again in the future. Secondly, if the unexpected result was enjoyable or intriguing, a new scheme will be established, incorporating the new attribute into an existing recognition pattern. For example, when children have an animal knowledge with their dog, the first time they see a deer, they might regard it as a dog because they cannot find any difference between dog and deer. However, once their parents say it is a deer because it has antlers that a dog does not have, the children will extend the animal scheme to include deer. In other words, the children can figure out differences and similarities between dog and deer, so they can do accommodation of their animal scheme as forming a new scheme of deer. In the same way, once fractions are given to children, they would apply the way of natural numbers to the fractional problems. However, they would figure out they are not the same by having incorrect answer, so they might try to find the different ways to fit in fractions. This process is learning fractions by extending their number scheme from natural numbers to rational numbers including fractions. Moreover, children would make a new fraction scheme.

Based on the reorganization hypothesis, instead of concentrating on interference from prior natural number knowledge, we believe educators should support students in building their understanding of fractions through accommodations in their own natural number knowledge. According to Steffe and Olive (2010), partitioning or splitting operations and iterable units in fractions knowledge can be incorporated into the same psychological framework as doing so in natural number knowledge. In other words, splitting and sequencing come from operation experiences in natural

numbers knowing.

## Students' Units Coordination

Composite units play an important role in understanding students' fractional knowledge. Thus, a way of generating and coordinating composite units helps students understand multiplicative concepts. In Hackenberg and Lee's (2015)2015 research, there are three types of multiplicative concepts: the *first* multiplicative concept (i.e., stage 1), the *second* multiplicative concept (i.e., stage 2), and the *third* multiplicative concept (i.e., stage 3). According to estimates made by Steffe (2007), at least two levels of interiorization were completed by 50–70% of newly enrolled sixth graders. That means there are more than half of students who are at stage 2 or stage 3 while there are over 30% of students at stage 1. In other words, teachers can see students with various levels of multiplicative concept, so they need to be prepared for what to do for each of them. We would like to discuss how teachers can help each level of students to understand fractions of knowledge.

Table 3 shows the percentage of students in grades 5-8 at each units coordination stage. Since students learn fractions in grades 5 and 6, and fractional knowledge requires multiplicative reasoning, the number of students at stage 1 highlights a challenge in constructing fraction knowledge (Hackenberg & Sevinc, 2024).

**Table 3.** Units coordination stages for grades 5-8, including percentages (Acar & Sevinç, 2021)

| Grade level | Stages of units coordination |            |           | Total      |
|-------------|------------------------------|------------|-----------|------------|
|             | Stage 1                      | Stage 2    | Stage 3   |            |
| 5           | 13 (52.0%)                   | 9 (36.0%)  | 3 (12.0%) | 25 (18.0%) |
| 6           | 10 (47.6%)                   | 10 (47.6%) | 1 (4.8%)  | 21 (15.1%) |
| 7           | 25 (48.1%)                   | 22 (42.3%) | 5 (9.6%)  | 52 (37.4%) |
| 8           | 14 (34.1%)                   | 25 (61.0%) | 2 (4.9%)  | 41 (29.5%) |
| Total       | 62 (44.6%)                   | 66 (47.5%) | 11 (7.9%) | 139 (100%) |

## Students operating at Stage 1

Students at stage 1 are able to coordinate two levels of units in an activity, but they are unable to accept a provided unit of units or composite unit (Hackenberg, 2013)2013. For example, stage 1 peers cannot imagine five parts in a unit ahead of operating. However, they can make five parts *in activity* by partitioning five parts from a whole unit. And they can find

three-fifths by shading three parts among five parts. In this regard, students at stage 1 have a tendency to construct *parts-within-wholes fraction schemes* (PWWFS) (Steffe & Olive, 2010). It brings the problem that fractions cannot go beyond a whole, so students at stage 1 typically struggle with constructing improper fractions.

In Olive & Vomvoridi's (2006) research, Tim was a student at stage 1. When he was given to solve ' $1/2 + 1/4$ ', he shaded one of two parts and all four parts and then added both regardless of the size of a part. So, he concluded ' $1/5$ ' as an answer. At first, he hadn't built a disembedding operation. Nevertheless, by combining focused interview exchanges with updated classroom instruction, over the course of a month Tim constructed a *partitive unit fraction scheme* (PUFS). Students with PUFS go beyond solely part-whole ideas (Hackenberg, 2013)2013. It shows some students at stage 1 can construct robust fractional knowledge with targeted intervention. Actually, they do not typically make the kind of development that Tim has. Students at stage 1 in other studies have continued to use the first multiplicative concept for at least two years despite receiving ongoing instruction from researchers with training (Steffe & Cobb, 1988).

## Students operating at Stage 2

For students at stage 2, it is possible to do units coordination as two composite units ahead of operating. For instance, students at stage 2 can imagine five parts of a unit partition, but they cannot partition a 5-unit bar into 7 parts in order to get a unit of 35 parts without activity. Thus, they can imagine a 5-unit bar without activity, and then they need to partition 7 parts in each of 5 parts. So, they can get 35 parts after the activity. Composing a unit and five parts in a unit means students have a second multiplicative concept (i.e., stage 2). And composing a unit, five parts in a unit, and seven parts in each five parts means students can think of three-levels-of-units concept which is not possible for students at stage 2. In other words, students at stage 2 can interiorize units of units by having a view of the result prior to the activity. Nevertheless, they are unable to project a three-level unit structure in the fraction bar.

In Hackenberg and Lee's (2015)2015 research, two students at stage 2 articulated verbally that drawing an improper fraction was strange. For example, Lisa said, "That's weird . . . can there be nine sevenths?" (p. 211) and added, "you can't take nine out of something that's seven" (p.211). She also drew nine parts less than a unit bar by partitioning seven parts with the small two last parts and dividing the first two parts equally in order to

make nine parts in a bar. Thus, the researchers did not attribute an iterative fraction scheme to Lisa. How is this related to what you have said above about students at stage 2?

### Students operating at Stage 3

For students at stage 3, it is possible to make a three-levels-of-units structure. Thus, students at stage 3 can conceive of 35 as seven 5s and as five 7s. That means students at stage 3 can imagine partitioning a 5-unit bar into 7 parts in order to get a unit of 35 parts without activity because they interiorized three-levels-of-units structure. It is powerful to construct any fractions and operate them.

In Hackenberg and Lee's (2015) 2015 research, Suzanne's drawing was representative of the responses of students at stage 3. In her drawing, she explained her reasoning as follows: "I divided this [original bar] up into sevenths, and then I drew out the whole bar, and then, I here [pointing to the right and of the copy of the whole bar] I kind of measured how much two sevenths would be, and added it to the end of the bar" (p.214). She also explained the size of one of the parts, Suzanne called it "one seventh of this bar [pointing to the original bar] or one ninth of this bar [pointing to the new bar]." She had therefore created a reversible iterative fraction method, according to the researchers. How does this relate to what you have said about students at stage 3?

## Partitive and Iterative Fraction Schemes

### Operations: Partitioning, Disembedding, and Iterating

*Operations* are mental actions, like conceiving of subdividing a unit into equal parts (Piaget, 1970; von Glasersfeld, 1995). For fractional knowledge, some key operations are *partitioning* of a unit, *disembedding* one of partitioning parts, and *iterating* as many as we need. For example, when students are given to draw seven-fifths, firstly they can draw a unit bar, and then they can partition it into five equal parts. Secondly, they need to disembed one-fifth part of a unit-bar. Finally, they iterate seven times of one-fifth in order to get seven-fifths. This process shows the operations of partitioning, disembedding, and iterating.

### The Partitive Fraction Scheme

Steffe (2001) determined that the first system that qualifies as a true fractional scheme is a *partitive fractional scheme*. A student who learns fraction as a part out of whole considers one-third of a candy bar as a part

of that. When  $\frac{1}{3}$  iterates three times, it will produce a three-parts bar. To make two-thirds of a candy bar, a student will divide into 3 parts of a candy bar first, pick one-third secondly, and then iterate it two times in order to make it as two-thirds. This implies that a unit fractional part of the partitioned whole can be disembed and then iterated to create another unit fractional part of the partitioned whole. However, enabling to disembed and iterate a unit fractional part does not mean those students are able to understand improper fractions such as four-thirds. In other words, these students cannot think out of the whole so they cannot conceive fractions that are greater than the unit.

### **The Iterative Fraction Schemes**

A significant challenge in students' fractional knowledge is an *iterative fractional scheme* (Olive & Steffe, 2002; Tzur, 1999) Steffe (2002). It starts from unit fractions as iterable units. If students have an iterative fractional scheme, they are able to find unit fractions and then make a whole by iterating the unit. For example, in order to find five-thirds, students who have an iterative fractional scheme can find one-third as a unit fraction, and then they do five times of one-third to get five-thirds. Moreover, they are enabled to consist whole by iterating three times of one-third (Hackenberg, 2007)2007.

### **Research Examples**

In Hackenberg's (2007)2007 research, Deborah did not show the construction of improper fractions initially. When she was given to draw seven-fifths by hand, she expressed some dismay. To be specific, Deborah stated that she drew the entire bar and didn't shut it until she had figured out how long two-fifths of the original bar were since she knew that seven-fifths equaled "one and two-fifths." Thus, Hackenberg (2007)2007 concluded Deborah gave examples of typical notions from pupils using partitive fractional schemes, and she deduced that seven-fifths did not yet mean one-fifth repeated seven times for them.

After 12 days, Deborah had another teaching episode about making a bar that was two-fifteenths longer than a  $\frac{13}{13}$ -bar. In this episode, Hackenberg used JavaBars which is useful to show splitting and iterating with correct size in a fraction bar. Deborah drew a unit bar with 15 pieces, took away a  $\frac{1}{15}$ -part from the  $\frac{15}{15}$ -bar and extended the bar by two-fifteenths of its original length. The researcher concluded that a bar two-fifteenths longer than a unit bar was both a whole unit bar and two-fifteenths more

and seventeen-fifteenths of the unit bar for Deborah since she justified her claim that seventeen-fifteenths is greater than one. Deborah's approach involved building a unit fraction as a unit that could be separated from and repeated beyond of the whole, creating a new unit of unit fractions that was still related to the entire but did not rely on part-whole connections. Thus, Hackenberg attributed an iterative fractional scheme to Deborah.

We think it is a strong strategy to partition a fraction bar and iterate a unit fraction by the given numerator, because student can recognize a unit fraction as the smallest part of a whole and make any number of fractions by iterating. Initially, when Deborah had improper fraction seven-fifth, she divided it by one and two-fifth before drawing. Therefore, she drew one bar and hesitated to close the fraction bar because she would like to make sure how much is two-fifth more. However, after having a notion of partitioning and iterating, she might partition as five pieces first, and then she will pull out one-fifth and iterate it seven times in order to make seven-fifth. This way is clear to express the corresponding size of seven-fifth.

### **Procedural Barriers in Fractions Learning: Toward Conceptual Understanding**

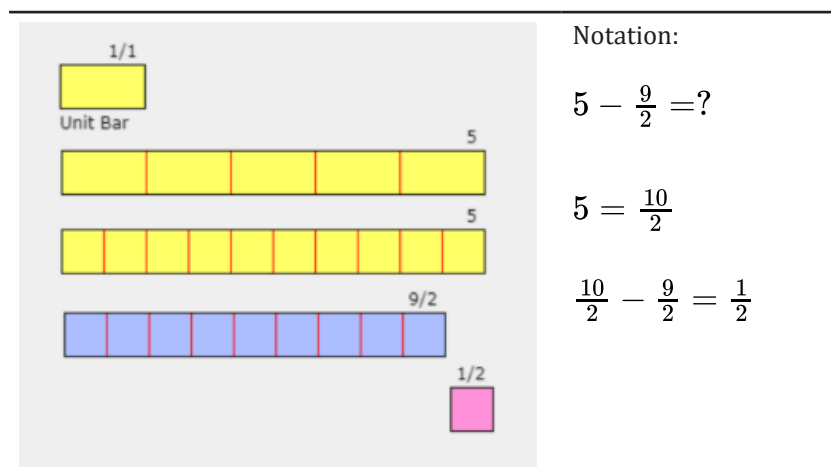
We have seen many teachers use procedure-based teaching style because they do not have enough time to do activities based on the allotted curriculum corresponding grades. In other words, teachers feel they have to address particular topics on particular days because there is pressure to "cover" topics. However, when students memorize the procedures as rules rather than understand by using visual methods and notations, they easily forget procedures in fraction operations such as addition, subtraction, multiplication, and division. We think if students need to remind how to operate fractions, using visualization examples with notation could be helpful for students to understand fractions knowledge better as well as remember it

### **From Rules to Reasoning: Using Drawing and Notation as Conceptual Tools**

In our N101 internship class, we were not accustomed to draw and explain fractional computations, so we were not able to understand the drawings immediately. Soon after writing notation by instructor Dr. Hackenberg, however, it was clear to understand procedure of units coordination, so we felt our fractional knowledge became more powerful and flexible to think of. The notation was word expression of the drawings regarding units

coordination. For example, notation ' $7/5 = 7 \times 1/5$ ' shows seven-fifth is equal to seven times one-fifth. It infers to partition a bar by five, disembed one-fifth as unit of unit, and iterate seven times in order to get seven-fifth. We think most of students who have learned fractional knowledge procedurally could be the same as me. If students can have the opportunity to learn fractions knowledge with drawing and notation, they would have more powerful fractions knowledge.

When the first author's 11<sup>th</sup> grade student in Ethiopia asked the reason why ' $5 - 9/2 = 1/2$ ', she just explained the rules of fractions operation procedurally. In this moment, she shared that she regrets her reaction for the question because it was about not reasoning mathematically but memorizing procedurally. If she would give to his visualization and notation together like in Figure 5, he might form more robust fractions scheme as well as remind fractions operation. When students start to drawings of quantities and build the notation from that, both young and older children, it can be helpful to understand fractions knowledge as well as make robust fractions knowledge.



**Figure 5.** Visual Fraction Bar with Drawing and Notation

### Fractions as Quantities, Not Symbols

Fractions can be thought of as measurable extensive quantities when students are required to illustrate fractional amounts in units of measurement using rectangles or segments (Hackenberg, 2010)2010. For example, if students can consider fractions as quantities, they can imagine the length of  $9/2$ cm and 5cm and compare them in order to subtract  $9/2$ cm from 5cm.

Furthermore, it is a noteworthy achievement when children see fractions as lengths instead of as parts of wholes (Steffe & Olive, 2010).

## **Conclusion**

In the first section, we emphasized fractions as a foundational concept in STEM education. Because all STEM areas involve numbers including fractions understanding fractions conceptually is essential for students. If students develop only a procedural understanding, this can hinder their ability to make sense of more advanced mathematics and STEM concepts. Conceptual understanding of fractions involves the ability to coordinate multiple levels of units and maintain this structure while working with them. We introduced this idea through the concept of units coordination.

Radical constructivism can serve as a lens for examining students' thinking in mathematics and across the broader STEM disciplines. This perspective holds that every student brings valuable experiences and can build new concepts upon their prior knowledge. For teachers, this underscores the importance of understanding students' thinking as a central part of instruction. We also discussed how this perspective can be implemented in classroom contexts by comparing it with the Emergent Perspective. Specifically, we presented our interpretation of Cobb and Yackel's work, first describing how the Emergent Perspective evolved from von Glasersfeld's Radical Constructivism, then explaining the limitations Cobb and Yackel identified in applying the theory to Developmental Research. We also noted how they incorporated aspects of Interactional Theory to address these limitations.

We examined the Emergent Perspective as an interpretive framework for Developmental Research. The constructs of Classroom Social Norms, Socio-mathematical Norms, and Classroom Mathematical Practices are key elements of the classroom micro-culture that Cobb and Yackel found important to account for. Each construct reflects individual aspects beliefs about one's own role, others' roles, and the nature of mathematical activity; mathematical beliefs and values; and mathematical conceptions and activity. These individual and social aspects cannot be fully understood in isolation they must be considered together.

As a final note on theory, we highlighted Cobb and Yackel's (1996) claim that the Emergent Perspective "locates students' mathematical development in social context while simultaneously treating learning as an active [individual] constructive process" (p. 173). The choice between a psychological perspective, a socio-cultural perspective, or an emergent

perspective depends on the specific research questions and goals. Sociocultural perspectives may be especially valuable for examining cultural diversity and reform, while a purely psychological perspective can be useful for deeply understanding individual learning processes.

From our case study on developing children's fraction knowledge, we found that understanding a student's true grasp of fractions means going beyond memorized rules to see how they approach and solve problems. Memorized rules can be forgotten, and their use does not necessarily indicate genuine learning. While state education policies, curricula, and textbooks may claim to support student-centered learning, in practice, instruction often defaults to rule-based teaching because it is faster and easier. In student-centered approaches, students should be guided to discover concepts themselves, rather than simply being told definitions and procedures. However, textbooks that give answers immediately after each question, or curricula that label themselves "constructivist" without truly supporting discovery, can hinder this process.

In our study, the student showed gaps in fraction concepts that should have been learned in primary school likely forgotten or never understood conceptually. We used Fraction Bars and the modeling method to help the student discover the meaning of multiplying fractions. While our one-month study cannot definitively prove lasting learning, the final interviews showed improvement in the student's problem-solving. We therefore recommend approaches that move away from rule-based teaching toward discovery-oriented methods. We also found that Fraction Bars are useful for supporting conceptual understanding, and that the modeling method is particularly effective for more complex fraction problems.

From the section on older students' fraction knowledge, we suggest four instructional goals for teachers: (1) Extend number domains: Guide students from natural numbers to integers and fractions, helping them connect and contrast these number types, (2) Develop fractional schemes: Use Steffe's operations (partitioning, disembedding, iterating) with units coordination to deepen understanding, including improper fractions, (3) Advance multiplicative concepts: Support students in moving from stage 2 to stage 3, emphasizing iterative fraction schemes, and (4) Use visualization and notation: Incorporate drawings and concise notation to strengthen conceptual understanding for students who have learned fractions procedurally.

Although this chapter focused on a specific mathematical concept, fractions, its implications extend to all STEM fields. Many STEM concepts, whether in physics, engineering, chemistry, or computer science, require interpreting and manipulating proportional relationships, ratios, and part whole reasoning skills rooted in fraction understanding. Strengthening these skills in mathematics can therefore directly benefit learning and problem-solving across STEM disciplines. Moreover, radical constructivism's emphasis on connecting new ideas to prior knowledge can guide teaching practices in any STEM subject, encouraging students to actively build understanding rather than passively receive information.

Additionally, we suggest four future research directions: (1) Longitudinal studies to track how conceptual fraction understanding in early grades impacts later STEM learning and career readiness, (2) Classroom-based experiments testing how tools like Fraction Bars and modeling methods affect learning outcomes in diverse student populations, (3) Cross-disciplinary studies exploring how fraction-related reasoning supports problem-solving in other STEM subjects, such as interpreting scientific data or scaling engineering designs, and (4) Investigations into teacher preparation programs to examine how constructivist principles are taught, modeled, and applied in practice. By continuing to explore these connections and strategies, educators and researchers can better prepare students not only to master fractions but also to engage deeply with the full range of mathematical and STEM challenges they will encounter throughout their education and beyond.

Ultimately, our work with Bernard reminds us that the heart of mathematics teaching lies in helping students make sense of what they do, not just in getting the right answer. When students are given the space to explore, represent, and connect ideas, they begin to see mathematics as something they can understand and even enjoy rather than as a set of rules to memorize. This shift in perspective can spark curiosity, build confidence, and lay a stronger foundation not only for future mathematics learning but for success across all STEM disciplines.

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